Exercise 87

An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

Solution

The velocity is the derivative of the position function.

$$\begin{aligned}
v(t) &= \frac{ds}{dt} \\
&= \frac{d}{dt}[Ae^{-ct}\cos(\omega t + \delta)] \\
&= \left[\frac{d}{dt}(Ae^{-ct})\right]\cos(\omega t + \delta) + Ae^{-ct}\left[\frac{d}{dt}\cos(\omega t + \delta)\right] \\
&= \left[(Ae^{-ct})\cdot\frac{d}{dt}(-ct)\right]\cos(\omega t + \delta) + Ae^{-ct}\left[-\sin(\omega t + \delta)\cdot\frac{d}{dt}(\omega t + \delta)\right] \\
&= \left[(Ae^{-ct})\cdot(-c)\right]\cos(\omega t + \delta) + Ae^{-ct}\left[-\sin(\omega t + \delta)\cdot(\omega)\right] \\
&= -cAe^{-ct}\cos(\omega t + \delta) - \omega Ae^{-ct}\sin(\omega t + \delta) \\
&= -Ae^{-ct}[c\cos(\omega t + \delta) + \omega\sin(\omega t + \delta)]
\end{aligned}$$

The acceleration is the derivative of the velocity function.

$$\begin{aligned}
a(t) &= \frac{dv}{dt} \\
&= \frac{d}{dt}\{-Ae^{-ct}[c\cos(\omega t + \delta) + \omega\sin(\omega t + \delta)]\} \\
&= \left[\frac{d}{dt}(-Ae^{-ct})\right][c\cos(\omega t + \delta) + \omega\sin(\omega t + \delta)] \\
&\qquad + (-Ae^{-ct})\left\{\frac{d}{dt}[c\cos(\omega t + \delta) + \omega\sin(\omega t + \delta)]\right\} \\
&= \left[(-Ae^{-ct})\cdot\frac{d}{dt}(-ct)\right][c\cos(\omega t + \delta) + \omega\sin(\omega t + \delta)] \\
&\qquad + (-Ae^{-ct})\left[-c\sin(\omega t + \delta)\cdot\frac{d}{dt}(\omega t + \delta) + \omega\cos(\omega t + \delta)\cdot\frac{d}{dt}(\omega t + \delta)\right] \\
&= [(-Ae^{-ct})\cdot(-c)][c\cos(\omega t + \delta) + \omega\sin(\omega t + \delta)] \\
&\qquad + (-Ae^{-ct})[-c\sin(\omega t + \delta)\cdot(\omega) + \omega\cos(\omega t + \delta)\cdot(\omega)]
\end{aligned}$$

Simplify the right side.

$$\begin{aligned} a(t) &= cAe^{-ct}[c\cos(\omega t+\delta)+\omega\sin(\omega t+\delta)] - Ae^{-ct}[-c\omega\sin(\omega t+\delta)+\omega^2\cos(\omega t+\delta)] \\ &= c^2Ae^{-ct}\cos(\omega t+\delta)+cA\omega e^{-ct}\sin(\omega t+\delta)+cA\omega e^{-ct}\sin(\omega t+\delta)-A\omega^2e^{-ct}\cos(\omega t+\delta) \\ &= A(c^2-\omega^2)e^{-ct}\cos(\omega t+\delta)+2cA\omega e^{-ct}\sin(\omega t+\delta) \\ &= Ae^{-ct}[(c^2-\omega^2)\cos(\omega t+\delta)+2c\omega\sin(\omega t+\delta)] \end{aligned}$$